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1996 J. Phys. A: Math. Gen. 29 263

(http://iopscience.iop.org/0305-4470/29/2/006)

J. Phys. A: Math. Gen. **29** (1996) 263–278. Printed in the UK

On the algebraic approach to cubic lattice Potts models

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Received 19 June 1995

Abstract. We consider diagram algebras, $D_{(G)}(Q)$ (generalized Temperley–Lieb algebras) defined for a large class of graphs G, including those of relevance for cubic lattice Potts models, and study their structure for generic Q. We find that these algebras are too large to play the precisely analogous role in three dimensions to that played by the Temperley–Lieb algebras for generic Q in the planar case. We outline measures to extract the quotient algebra that would illuminate the physics of three-dimensional Potts models.

lines

1. Introduction

With the benefit of hindsight it is striking how easy it might have been, 15–20 years ago, to identify roots of unity as the values of q that were special for the description of the physics of $Q = (q + q^{-1})^2$ state Potts models in two dimensions, and related spin chains in one dimension. It is the work of a few lines to derive these as the exceptional cases using the Temperley–Lieb algebra introduced by Temperley and Lieb (1971) [1] (see [9]). This could have been done before many of the models were solved. Only the interpretation of this result might have puzzled the early 'algebraic physicist'. Of course, this is not the way things happened. The location of the special points is revealed in the *details* of the solution of the models [2, 3, 7], and it was only after the solution of the models that the significance of the special points and their relation to the cataloguing of models into universality classes was appreciated.

In a sense, we find ourselves heading down the same path now for three and higher dimensional models. There has been some very impressive work done on models whose Boltzmann weights satisfy the tetrahedron equations [8], but that is not the route we follow here. In [20] it was suggested that the diagram algebras $D_G(Q)$ (defined below) for some sequence of graphs $G^{(-)} = \{G^{(1)}, G^{(2)}, \ldots\}$ would play the role of the Temperley–Lieb algebra for higher dimensions (the Temperley–Lieb algebra is the sequence of diagram algebras with $G^{(j)} = A_j$, where A_j is the j-node chain graph). In this paper we determine the structure of $D_G(Q)$ for enough graphs G to show that a direct analogy with two dimensions is too simplistic in general, and suggest a resolution.

The paper is structured in the following way. We introduce the Q-state Potts model on any lattice, and point out the relation between the transfer matrix of the two-dimensional model and the Temperley–Lieb (TL) algebra. Since we take the algebraic route in this paper, we then state the specific link between representation theory (the index set for distinct irreducible representations) and physics (primary fields in the two-dimensional conformal field theory (CFT)) that we would like to examine in the higher dimensional context. Namely,

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when the index set is *finite*, the corresponding CFT is *minimal*. In two dimensions, the index set is finite at the special values of Q called Beraha numbers, which are also the values at which the TL algebras defined for a sequence of chain graphs of increasing length become non-semisimple beyond some length. One of our objectives in this paper is to locate the corresponding Q-values at which our candidate algebra $D_G(Q)$ becomes non-semisimple in an analogous way.

We define the diagram algebra as a subalgebra of the partition algebra [19] in the last part of this section. The basis of the defining representation of the diagram algebra is taken from the set of partitions of the nodes of two copies of a graph G, called 'top' and 'bottom'. Multiplication in the algebra involves stacking one such top and bottom over another, and keeping track of the resulting partitions by transitivity (see figure 1). In section 2, we tackle the problem of classifying the irreducible representations of $D_G(Q)$ for generic Q. This is carried out in two steps—first by noting the number of parts with both top and bottom nodes as above (called the number of 'propagating lines') and then by the permutations of these lines allowed on a given graph G. We do this for a large class of graphs and, in particular, for a class of graphs which we call *unsplitting* (see proposition 3 and the remark following it). We also give necessary and sufficient conditions for a set of partitions to be a basis for these irreducibles in proposition 6. Using this key result, we prove in proposition 7 that the algebras defined for a sequence of unsplitting graphs ceases to be semi-simple for at least all integer values of Q. In section 4, we apply the above results for the particular example of an unsplitting graph that is relevant for building the transfer matrix of the three-dimensional Potts model. We discuss the implication of these results next. The appendix lays out the preliminary steps towards the description of the Bratteli diagram (or the inclusion matrix) for the restriction of modules for the generically semi-simple algebras $D_H \subset D_G$ for graphs G, H and $H \subset G$.

1.1. Basic definitions

For any simple, unoriented graph L, and natural number Q, the partition function of the Q-state Potts model [4] on the graph L is

$$Z(L) = \sum_{\substack{\sigma_i \in \{1,2,\ldots,Q\} \\ \forall i \in \Lambda^0_L}} \exp\left(\beta \sum_{(i,j)\in\Lambda^1_L} \delta_{\sigma_i\sigma_j}\right) \tag{1}$$

where Λ^0_L denotes the set of nodes of L, and Λ^1_L, the set of its edges.

Recall that for graphs G and H then $G \times H$ is a graph such that

$$\Lambda^0_{G\times H} = \Lambda^0_G \times \Lambda^0_H \tag{2}$$

and

$$\big((i,j),(k,l)\big) \in \Lambda^1_{G\times H} \qquad \text{if} \begin{cases} (i,k) \in \Lambda^1_G & \text{and} \quad j = l \quad \text{or} \\ (j,l) \in \Lambda^1_H & \text{and} \quad i = k\,. \end{cases} \tag{3}$$

Let $\hat{A}_t$ be the t-node closed chain graph. Then for example $A_l \times A_m \times \hat{A}_t$ would be the cubic lattice with periodicity in one direction. For any G the partition function

$$Z(G \times \hat{A}_t) = \mathrm{Tr}\big((\tau_G)^t\big) \tag{4}$$

where τ_G is the (G shaped layer) transfer matrix defined as

$$\tau_G = \prod_{i\in\Lambda^0_G} \left((e^\beta - 1)\mathbf{I} + \sqrt{Q}U_{i\cdot}\right) \prod_{(i,j)\in\Lambda^1_G} \left(\mathbf{I} + \frac{(e^\beta-1)}{\sqrt{Q}} U_{i,j}\right). \tag{5}$$

Here

$$\mathbf{I} = \mathbf{I}_Q \otimes \mathbf{I}_Q \otimes \ldots \otimes \mathbf{I}_Q \tag{6}$$

(one factor for each node of G, each factor a $Q \times Q$ unit matrix)

$$U_{i\cdot} = \frac{1}{\sqrt{Q}} \left(\mathbf{I}_Q \otimes \mathbf{I}_Q \otimes \ldots \otimes \underbrace{M}_{i\text{th}} \otimes \ldots \mathbf{I}_Q\right) \qquad (i \in \Lambda^0_G) \tag{7}$$

where M is the $Q \times Q$ matrix with all entries 1, in the ith position (note that writing the factors in a row implies a total order on Λ^0_G—this is physically misleading for general G and can be chosen arbitrarily, cf the two-dimensional case [2]) and

$$U_{i,j} = \sqrt{Q}\left(\mathbf{I}_Q \otimes \mathbf{I}_Q \otimes \ldots \otimes \underbrace{N}_{i\text{th}\otimes j\text{th}} \otimes \ldots \mathbf{I}_Q\right) \qquad \left((i,j) \in \Lambda^1_G\right) \tag{8}$$

where N is the $Q^2 \times Q^2$ diagonal matrix acting on the ith and jth subspaces (and note that j is not necessarily adjacent to i in a given ordering) with index set $\{1, 2, \ldots, Q\} \times \{1, 2, \ldots, Q\}$, and

$$N_{(i,j),(i,j)} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

(see [1, 2, 23]).

Note that these matrices obey

$$U^2_{i\cdot} = \sqrt{Q}U_{i\cdot} \qquad U^2_{i,j} = \sqrt{Q}U_{i,j} \qquad U_{i\cdot}U_{i,j}U_{i\cdot} = U_{i\cdot} \qquad U_{i,j}U_{i\cdot}U_{i,j} = U_{i,j} \tag{10}$$

$$[U_{i\cdot}, U_{j\cdot}] = [U_{i\cdot}, U_{j,k}] = [U_{i,j}, U_{k,l}] = 0 \qquad i \neq j, k\,. \tag{11}$$

Recall that for $G = A_n$ the graph $L = G \times \hat{A}_t$ is the square lattice on a cylinder, and these matrices give a representation of the Temperley–Lieb algebra [1]. It is known that this representation is faithful except at the Beraha-type numbers [16] $Q = 4\cos^2 \pi p/b$ (p, b integers), where it is faithful only on the unitarizable quotient [15]. Also, for other Q values the number of distinct irreducible representations in this Potts representation grows unboundedly with n, whereas for p, b integer it is finite and fixed by b (*à la* primary fields in rational conformal field theories [11]). The models corresponding to these Beraha-type numbers have as massless Euclidean field theory limits the minimal models of conformal field theory. For $p = 1$, these lattice models are in the same universality class as the ABF models [12, 5, 14, 6] whose corresponding conformal field theories belong to the unitary series of [13] with $c = 1 - 6/[b(b-1)]$.

In this paper we address the question of what is the appropriate abstract algebra, in the same sense as above, for the arbitrary sequence $G^{(-)}$. In [20], it has been noted that the algebra with generators and relations simply as in equation (11) (the full Temperley–Lieb algebra) is too big, as the Potts representation is then never faithful for non-chain graphs. Instead, we shall focus on the following finite dimensional quotients. In order to define these quotients, it is useful to recall the definition of the partition algebra $P_n = P_n(Q)$ [19, 20].

Let $\boldsymbol{S}_{2n}$ be the set of partitions of the set $\{1, 2, \ldots, n, 1', 2', \ldots, n'\}$. The $\mathbb{C}$-linear extension of the product defined in figure 1 on the vector space with basis $\boldsymbol{S}_{2n}$ gives the partition algebra, $P_n(Q)$.

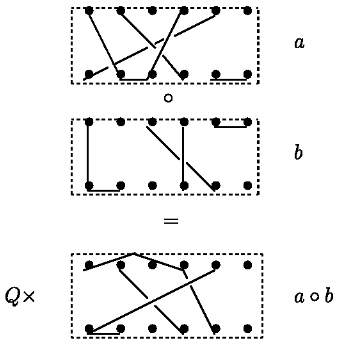

Figure 1. The top diagram is a, the one in the middle b, and the one at the bottom is the product $a \circ b$. Trace the connectivities from bottom to top, and for each discarded part from the middle, pick up a factor of Q to obtain $a \circ b$.

The diagram algebra, $D_G(Q)$, for a graph G is defined as the subalgebra of the partition algebra with generators:

$$\begin{aligned} 1 &= \big((11')(22')\dots(nn')\big) \\ A^{i\cdot} &= \big((11')(22')\dots(i)(i')\dots(nn')\big) \qquad \forall i \in \Lambda_G^0 \\ A^{i,j} &= \big((11')(22')\dots(i\ j\ i'\,j')\dots(nn')\big) \qquad \forall (i,j) \in \Lambda_G^1 . \end{aligned} \tag{12}$$

Note that $1^{ij} = \big((11')(22')\dots(ij')\dots(ji')\dots(nn')\big) \in P_n(Q)$, is *not* in $D_G(Q)$.

The diagram algebra may be also be thought of (visualized) on $G \times A_k$ (k large) as the restriction of $P_n(Q)$ to partitions achievable as connectivities (i.e. a set of mutually non-intersecting trees, cf [10]) between the nodes of the bottom layer (the nodes $(i,1)$ to be called i $\forall i \in \Lambda_G^0$), and those of the top layer (the nodes (i,k) to be called $i' \forall i \in \Lambda_G^0$).

Note that, with $V = \mathbb{C}^Q$,

$$\rho_G : D_G(Q) \longrightarrow \mathrm{End}(V^{\otimes|\Lambda_G^0|}) \tag{13}$$

given by

$$\rho_G(A^{i\cdot}) = \sqrt{Q}U_{i\cdot} \qquad \rho_G(A^{i,j}) = \frac{1}{\sqrt{Q}}U_{i,j} \tag{14}$$

is a representation of the diagram algebra called the Potts representation (equations (7), (8)).

The Potts representation is generically faithful for $G = A_n$, and for this reason, here we try $D_G(Q)$ as a candidate for the appropriate generalization of the Temperley–Lieb algebra for arbitrary graph G. Note, in particular, that $D_{A_n}(Q)$ is isomorphic to the Temperley–Lieb algebra for any Q, including non-integer values.

The partition function $Z(L)$ may be computed working in $D_G(Q)$ instead of in the defining Potts representation [2], as in the two-dimensional case, where the $D_{A_n}(Q)$ calculation is that of the square lattice dichromatic polynomial [3, 10].

In the two-dimensional case the exceptional models may be identified directly at the level of algebra by finding the Q values for which the structure of the Temperley–Lieb algebra departs from the generic semi-simple structure. Our idea is that the departures from generic behaviour would be important for arbitrary G. The structure of $D_G(Q)$ is important 'physically', since it may be used to characterize the spectrum of the transfer matrix, $\mathcal{T}_G$. Thus we proceed to analyse the structure of $D_G(Q)$. This is already known for some G; in particular, for $G = A_n$ and for $G = K_n$, the complete graph on n nodes [19]. In this paper we consider graphs appropriate for higher dimensional Potts models and dichromatic polynomials, including sequences appropriate for the physically crucial cubic lattice Potts models, and the bi-plane lattices to which recent ideas in high-T_c superconductivity have drawn attention [17].

2. Generic structure of $D_G(Q)$

Mathematically, the first step in determining the structure (representation theory) of an algebra is generally to label the irreducible representations. In what follows we take $n = |\Lambda_G^0|$. The irreducible representations of $P_n(Q)$ are labelled by

$$\mathcal{L}_n = \{\lambda \vdash i : i = 0, 1, 2, \ldots, n\}$$

and since $D_G(Q) \subset P_n(Q)$ all the irreducibles must be somehow contained in the irreducibles of $P_n(Q)$.

Consider $P_n(Q)$ as a $D_G(Q)$ module. Clearly any $P_n(Q)$ module is also a $D_G(Q)$ module. Now $P_n(Q)$ has been filtered into invariant subspaces with bases

$$\boldsymbol{B}_i = \{x \in \boldsymbol{S}_{2n} | \#^p(x) \leqslant i\}$$

where $\#^p(x)$ is the number of parts of x containing both primed and unprimed nodes, called the 'propagating number' of x.

If we define

$$\boldsymbol{E}_i{}^{(n)} = \prod_{j=i+1}^{n} \left(\frac{A^{j\cdot}}{Q} \right)$$

then $\mathbb{C}$-span$(\boldsymbol{B}_i) = P_n \boldsymbol{E}_i P_n$. For a given n, we drop the superscript (n) and write $\boldsymbol{E}_i$. Note that $\#^p(\boldsymbol{E}_i) = i$ and for $a, b \in \boldsymbol{S}_{2n}$,

$$\#^p(ab) \leqslant \min(\#^p(a), \#^p(b))$$

and we ignore elements of $\mathbb{C}$ in evaluating $\#^p(z)\ \forall z \in P_n(Q)$. Thus

$$P_n[i] = P_n \boldsymbol{E}_i P_n / P_n \boldsymbol{E}_{i-1} P_n$$

is a $P_n(Q)$ module with basis $\boldsymbol{B}_i \setminus \boldsymbol{B}_{i-1}$. Note that in the diagrammatic realization of the left action of $D_G(Q)$ on $P_n[i]$ the 'bottom' of each $x \in \boldsymbol{B}_i \setminus \boldsymbol{B}_{i-1}$ (i.e. the connectivities of the unprimed nodes of any $x \in \boldsymbol{S}_{2n}$) remains unchanged. That is, all elements with the same bottom form a submodule.

For example, $\Delta_i := P_n \boldsymbol{E}_i$ (mod $P_n \boldsymbol{E}_{i-1} P_n$) is one of the left P_n submodules of $P_n[i]$, and $P_n[i]$ may be decomposed into submodules all of which are isomorphic. Note that Δ_i has a basis the set of partitions which have each unprimed node in a different part, the last $n - i$ nodes singletons (i.e. in parts on their own), the others connected to primed nodes.

In fact, as a left $D_G(Q)$ module Δ_i breaks as $D_G(Q)\boldsymbol{E}_i \oplus R_i$ where R_i is either empty or a direct sum of one-dimensional modules (see the appendix), so we need only focus on $D_G \boldsymbol{E}_i$ mod $D_G \boldsymbol{E}_{i-1} D_G$.

The final piece of the jigsaw for $P_n(Q)$ is to note that $P_n \boldsymbol{E}_i$ is a projective right $\mathsf{S}(i)$ module (i.e. a direct summand of a direct sum of copies of the regular representation of the symmetric group [18]) where the action is to permute the first i (unprimed) nodes. For example, see figure 2. Thus $P_n \boldsymbol{E}_i$ (mod) breaks up into simple modules indexed by $\lambda \vdash i$ (from $\mathsf{S}(i)$ representation theory [21]).

For $D_G(Q)$, however, the picture is more complicated, since $D_G \boldsymbol{E}_i$ is not always closed under the right action of $\mathsf{S}(i)$. For example, whereas $P_n(Q)\boldsymbol{E}_n \cong \mathsf{S}(n)$ modulo $P_n(Q)\boldsymbol{E}_{n-1} P_n(Q)$, we have $D_G \boldsymbol{E}_n = \mathbb{C}\boldsymbol{E}_n = \mathbb{C} \cdot 1$ mod $D_G \boldsymbol{E}_{n-1} D_G$ for any G. To see this note that with n propagating lines from bottom to top of $G \times A_k$ (k large) there is only one possibility, as depicted in figure 3. Our problem is thus reduced to determining the maximum subgroup $H_G^i \subset \mathsf{S}(i)$ for which $D_G \boldsymbol{E}_i$ (mod $D_G \boldsymbol{E}_{i-1} D_G$) is a right module. In general, for $i < n$, the situation depends on G.

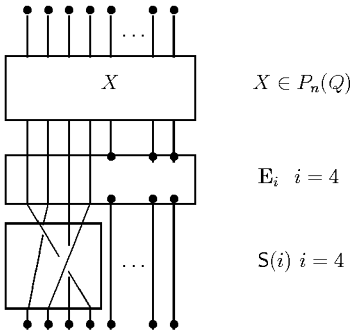

Figure 2. The right action of the permutation group (depicted in the box) is, diagrammatically, the action from below.

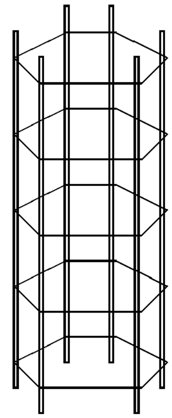

Figure 3. There is no space for lateral motion if all of the nodes of G (the hexagon) are propagating. The propagating lines are drawn with double lines.

Before actually determining H_G^i, let us first explicitly construct words in the algebra that would implement the group action. As is clear from figure 3, one or more nodes of G need to be disconnected to allow for walks on $G \times A_k$ to realize any permutations of the nodes (except for the identity permutation as in figure 3). Also, since there is no unique, or natural ordering of the nodes of G, we need to determine whether H_G^i depends on the choice of the nodes disconnected by $\boldsymbol{E}_i$.

For any subset, s of $\{1, 2, \dots, n\}$, let p_s be the set difference $\{1, 2, \dots, n\} \setminus s$, and $\boldsymbol{E}_{\{p_s\}} = \prod_{j \in s} (A^{j\cdot}/Q)$. For example, for $\underline{i} = \{i+1, i+2, \dots, n\}$,

$$p_{\underline{i}} = \{1, 2, \dots, i\} \qquad \text{and} \qquad \boldsymbol{E}_{\{p_{\underline{i}}\}} = \prod_{j=i+1}^{n} \left(\frac{A^{j\cdot}}{Q} \right) = \boldsymbol{E}_i \,. \tag{15}$$

Definition 1. The partition basis of $D_G(Q)$, denoted by $\boldsymbol{S}_{2n}^G$ is $\boldsymbol{S}_{2n} \cap D_G$. Also, set $\boldsymbol{B}_i{}^G := \boldsymbol{B}_i \cap D_G$.

Definition 2. Let $s, t \subset \{1, 2, \dots, n\}$, s.t. $|s| = |t|$. Then

$$\Phi_s^t := \{\varphi_s^t | \varphi_s^t = \boldsymbol{E}_{\{p_t\}} X \boldsymbol{E}_{\{p_s\}} \quad \forall X \in \boldsymbol{S}_{2n}^G \qquad \text{s.t. } \#^p(\boldsymbol{E}_{\{p_t\}} X \boldsymbol{E}_{\{p_s\}}) = |p_s|\} \,. \tag{16}$$

These elements of (16) may be interpreted as bijections, $\varphi_s^t : p_s \to p_t$. Note, in particular, that $\Phi_s^s \subseteq \mathbf{S}(p_s)\boldsymbol{E}_{\{p_s\}}$.

Let $\delta = s \triangle t = (s \setminus t) \cup (t \setminus s)$, the symmetric difference of sets s, t, with $|\delta| = 2d$, i.e. d elements of the $n - i$ elements of s are distinct from those of t. Then $\exists$ partitions of δ of shape 2^d, i.e. of the form

$$\big((\delta_1\delta_1^*)(\delta_2\delta_2^*)\dots(\delta_d\delta_d^*)\big) \tag{17}$$

where the unstarred nodes of $\delta \in s$ and the starred ones in t. Consider chain subgraphs, $A_{\mu_i}^{(i)}, i = 1, \dots, d$ of the connected graph G, with nodes labelled $x_j^{(\mu_i)}$, $j = 1, 2 \dots, \mu_i$ such that the first node of $A_{\mu_i}^{(i)}$ is $x_1^{(\mu_i)} = \delta_i$ and the μ_ith, $x_{\mu_i}^{(\mu_i)} = \delta_i^*$. Let us construct words $\omega_{A_{\mu_i}^{(i)}}$ of the form

$$\omega_{A_{\mu_i}^{(i)}} = \prod_{j=0}^{\mu_i - 2} \big(A^{x_{\mu_i - j}^{(\mu_i)}\cdot} \; A^{x_{\mu_i - j}^{(\mu_i)}, x_{\mu_i - j - 1}^{(\mu_i)}}\big) A^{x_1^{(\mu_i)}\cdot} \,. \tag{18}$$

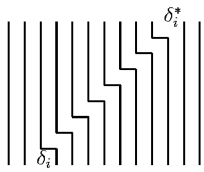

Figure 4. The element of the algebra shifting the 'hole' from δ_i to δ_i^*. Note the minimum height required to achieve this connectivity is of the order of the distance $|\delta_i - \delta_i^*|$.

s.t. $\omega_{A^{(i)}_{\mu_i}}$ achieves the connectivity which differs from the unit in

$$\left(\cdots({}_{,}x_1^{(\mu_i)})({}_{,}x_2^{(\mu_i)}x_1^{(\mu_i)\prime})({}_{,}x_3^{(\mu_i)}x_2^{(\mu_i)\prime})\ldots({}_{,}x_{\mu_i}^{(\mu_i)}x_{\mu_i-1}^{(\mu_i)\ \prime})(x_{\mu_i}^{(\mu_i)\prime})\ldots\right) \in D_G(Q) \tag{19}$$

on the sublattice $A^{(i)}_{\mu_i} \times A_k$, $(k > \mu_i)$, where (as before) $(x_j^{(\mu_i)}, 1) = {}_{,}x_j^{(\mu_i)}$ and $(x_j^{(\mu_i)}, k) = x_j^{(\mu_i)\prime}$ (see figure 4).

It is useful to view the element of the algebra as one that pushes a 'hole' from its location in s to one in t. The equivalence relation that defines the algebra $D_G(Q) \subset P_n(Q)$ implies that

$$\omega_{\{\mu_i\}} = \prod_{i=1}^{d} \omega_{A_{\mu_i}} \in \Phi_s^t \tag{20}$$

independent of the choice of graphs $A^{(i)}_{\mu_i}$ connecting the nodes of δ. Thus,

Proposition 1.

$$D_G \boldsymbol{E}_{\{p_s\}} D_G = D_G \boldsymbol{E}_i D_G \quad \forall s \subset \{1, 2, \ldots, n\} \qquad \text{s.t.} \quad |s| = n - i\,. \tag{21}$$

The different choices of pairing the starred and unstarred nodes of δ give different bijections $\varphi_s^t \in \Phi_s^t$. Let $H_G^{p_s} := \Phi_s^s$. We then have

Proposition 2. For any fixed element $\varphi_t^s \in \Phi_t^s$, $\varphi_t^s \Phi_s^t = H_G^{p_s}$ and $H_G^{p_s} = H_G^{p_t}$ if $|s| = |t|$.

Proof. For sets s, t, r of the same cardinality, these 'bijections' obey

$$\rho_t^r \circ \varphi_s^t \in \Phi_s^r \qquad \forall \rho_t^r \in \Phi_t^r \qquad \text{and} \qquad \varphi_s^t \in \Phi_s^t\,. \tag{22}$$

Therefore,

$$\Phi_t^r \Phi_s^t \subseteq \Phi_s^r \Rightarrow |\Phi_t^r| \leqslant |\Phi_s^r| \qquad \text{and} \qquad |\Phi_s^t| \leqslant |\Phi_s^r| \Rightarrow |\Phi_s^r| = |\Phi_t^u| \tag{23}$$

for any $r, s, t, u \in \{1, 2, \ldots, n\}$ with $|r| = |s| = |t| = |u|$.

In particular, $|\Phi_t^s| = |\Phi_s^s|$ and $\varphi_t^s \Phi_s^t \subseteq \Phi_s^s \Rightarrow \varphi_t^s \Phi_s^t = \Phi_s^s$, for a fixed $\varphi_t^s \Phi_t^s$. Also, $\rho_s^t \Phi_s^s \varphi_t^s = \Phi_t^t$. This implies that $\Phi_s^s \cong \Phi_t^t$ and depends only on the cardinality, $|p_s|$. $\square$

Corollary 2.1.

$$D_G \boldsymbol{E}_{\{p_s\}} \cong D_G \boldsymbol{E}_i \quad \text{and} \quad \boldsymbol{E}_i D_G \boldsymbol{E}_i \cong \boldsymbol{E}_i \otimes H_G^i \qquad \text{mod} \ \ D_G \boldsymbol{E}_{i-1} D_G \tag{24}$$

where H_G^{i-1} is no smaller than the maximal subgroup of $\mathbf{S}(i-1)$ contained in H_G^i (so that in particular if any $H_G^i = \mathbf{S}(i)$ then $H_G^j = \mathbf{S}(j)\ \forall j < i$).

Hence,

Theorem 1. Let Γ_G^i, Γ_G be index sets for irreducible representations of H_G^i and $D_G(Q)$ respectively. Then

$$\Gamma_G = \cup_i \Gamma_G^i\,.$$

2.1. How to compute H^i_G

For an arbitrary graph G, $|\Lambda^0_G| = n$, and $\alpha \in \Lambda^0_G$, consider closed chain subgraphs, $\hat{A}_{p+1} \subset G$ with $\alpha \in \Lambda^0_{\hat{A}_{p+1}}$. Setting $d = 1$, $x_1^{(\mu_i)} = x_{\mu_i}^{(\mu_i)}$ and $\mu_i = p + 2$ in (20), we note $\Phi^{\{\alpha\}}_{\{\alpha\}}$ contains $\mathbf{Z}_p$. By pushing the hole around, by (18), so as to lie on other closed chain subgraphs, the set of these p-cycles generates H^{n-1}_G.

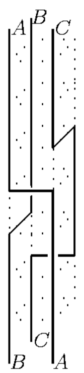

Figure 5. An example of a 3-cycle (BAC), with $G = A_2 \times A_2$. If we label the nodes such that at the bottom layer, A, B and C are drawn through 1, 2 and 3 respectively, with the 'hole' at 4, $p_{\{4\}} = \{1,2,3\}$ and the connectivity drawn is $((12')(23')(31')) \in \Phi^{\{4\}}_{\{4\}}$.

For any graph G, let $G^\times_x$ denote the graph obtained by removing the node $x \in \Lambda^0_G$ and the bonds connected to it, i.e. $\Lambda^0_{G^\times_x} = \{1, 2, \ldots, n\} \setminus \{x\}$ and $\Lambda^1_{G^\times_x} = \Lambda^1_G \setminus \{(i, x)|(i, x) \in \Lambda^1_G\}$. Let $G^\times{}_{(x,y)}$ denote the graph obtained by removing the bond (x, y), i.e., $\Lambda^0_{G^\times{}_{(x,y)}} = \Lambda^0_G$ and $\Lambda^1_G \setminus \Lambda^1_{G^\times{}_{(x,y)}} = \{(x, y)\}$. Recall, $\boldsymbol{E}_{n-1} D_G \boldsymbol{E}_{n-1} \cong \boldsymbol{E}_{n-1} \otimes H^{n-1}_G$. Then, we have

Proposition 3. Let $|\Lambda^0_{G_i}| = n_i \forall i$.

(i) If $G^\times{}_{(x,y)} = G_1 \sqcup G_2$, then $H^{n-1}_G = H^{n_1-1}_{G_1} \times H^{n_2-1}_{G_2}$ where the two factors act on the $n_1 - 1$ and $n_2 - 1$ nodes in G_1 and G_2, respectively.

(ii) If $G^\times_x = G'_1 \sqcup G'_2 \sqcup \cdots$, then $\exists G_i \supset G'_i \forall i$ s.t. $\cap_i \Lambda^0_{G_i} = \{x\}$ and $\cap_i \Lambda^1_{G_i} = \{\emptyset\}$, s.t. $H^{n-1}_G = H^{n_1-1}_{G_1} \times H^{n_2-1}_{G_2} \times \cdots$, where $H^{n_i-1}_{G_i}$ acts on G_i only.

Remark. Let us call graphs G that do not decompose in the sense of the previous proposition *unsplitting*. A simple example of an unsplitting graph is the closed chain graph $\hat{A}_n$. We shall consider other examples below.

Definition 3. The graph $\Theta^n_{p,q} (q < p \leqslant n - 1 < 2p + q)$ has n nodes labelled $\{1, 2, \ldots, n\}$, and bonds,

$$\Lambda^1_{\Theta^n_{p,q}} := \left\{ \begin{array}{l} (1, n),\ (p + q, n),\ (p, p + q + 1), \\ (i\ i + 1);\ i = \{1, 2, \ldots, n - 1\} \setminus \{p + q\}\,. \end{array} \right\}$$

(see figure 6). Note that $\Theta^n_{n-1,0} = \hat{A}_n$.

Proposition 4. All unsplitting graphs G that are not closed chain graphs contain $\Theta^n_{p,q}$ as a subgraph for some positive integers p, q.

Proposition 5. For $G = \Theta^n_{p,q}$, let $\beta_1 := \{p + q + 1, p + q + 2, \ldots, n - 1\}$, $\beta_2 := \{1, 2, \ldots, p - 1\}$ and $\beta_3 := \{p + 1, p + 2, \ldots, p + q\}$, and let $\alpha_i := \{1, 2, \ldots, n\} \setminus \beta_i$, $i = 1, 2, 3$. Then, for $a_i \in \alpha_i$, $i = 1, 2, 3$, and defining $\Phi^{\{a_i\}}_{\alpha_i \{a_i\}} \subseteq \Phi^{\{a_i\}}_{\{a_i\}}$ to be the words of the form $\omega_{A_{\alpha_i}}$ as in (18), we have $\Phi^{\{a_1\}}_{\alpha_1 \{a_1\}} \cong \mathbf{Z}_{p+q}$, $\Phi^{\{a_2\}}_{\alpha_2 \{a_2\}} \cong \mathbf{Z}_{n-p}$ and $\Phi^{\{a_3\}}_{\alpha_3 \{a_3\}} \cong \mathbf{Z}_{n-q-1}$. Also $\varphi^{\{a_i\}}_{\{a_i\}}(x) = x$, $\forall \varphi^{\{a_i\}}_{\{a_i\}} \in \Phi^{\{a_i\}}_{\alpha_i \{a_i\}}$ and $x \in \beta_i$ for $i = 1, 2, 3$.

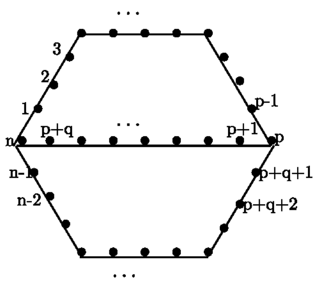

Figure 6. The graph $\Theta^n_{p,q}$ defined above.

Corollary 5.1. For $G = \Theta^n_{p,q}$, $\alpha \subset \Lambda^0_G$, $|\alpha| \geqslant 1$, $\exists \varphi^\alpha_\alpha \in \Phi^\alpha_\alpha$ such that for $x, y \in \Lambda^0_G \setminus \alpha$, $\varphi^\alpha_\alpha(x) = 1$ and $\varphi^\alpha_\alpha(y) = n$.

Proof. This is achieved by a sequence of words as in the proposition that 'moves' one of x, y, while keeping the other fixed. Such a move is easiest if $x \in \alpha_i$ and $y \in \alpha_j$, $i \neq j$. If x, $y \in \alpha_i$, we first move both until only one of x, y is in β_j for some j. □

Corollary 5.2. For $\Theta^n_{p,q}$ as above, and $n \geqslant 4$,

$$H^{n-1}_{\Theta^n_{n-2,0}} = \mathbf{S}(n-1)$$

$$H^{n-1}_{\Theta^n_{n-3,0}} = \begin{cases} \mathbf{S}(n-1) & \text{for} \quad n \text{ odd} \\ \mathbf{A}(n-1) & \text{for} \quad n \text{ even}. \end{cases} \tag{25}$$

$$H^{n-1}_{\Theta^n_{n-3,1}} = \begin{cases} \mathbf{A}(n-1) & \text{for} \quad n \text{ odd} \\ \mathbf{S}(n-1) & \text{for} \quad n \text{ even}. \end{cases}$$

Proof. By the same procedure as in the corollary above, it is possible to construct $\varphi^{\{n\}}_{\{n\}} \in \Phi^{\{n\}}_{\{n\}}$, such that for $G = \Theta^n_{n-2,0}$, $\varphi^{\{n\}}_{\{n\}}(x) = n-2$ and $\varphi^{\{n\}}_{\{n\}}(y) = n-1$ for $x, y \in \Lambda^0_G \setminus \{n\}$ and $G = \Theta^n_{n-3,0}, \Theta^n_{n-3,1}$, $\varphi^{\{n\}}_{\{n\}}(x_i) = n - i$, $i = 1, 2, 3$ and $x_i \in \Lambda^0_G \setminus \{n\}$.

Thus, for $\Theta^n_{n-2,0}$ we can achieve arbitrary transpositions, which generate $\mathbf{S}(n-1)$, and for $\Theta^n_{n-3,0}, \Theta^n_{n-3,1}$ we can achieve all 3-cycles, generating $\mathbf{A}(n-1)$. However, by the proposition above, we can also realize $\Phi^{\{a_1\}}_{\alpha_1\{a_1\}} \cong \mathbf{Z}_{p+q}$, which for $G = \Theta^n_{n-3,0}, \Theta^n_{n-3,1}$ are $\mathbf{Z}_{n-3}$ and $\mathbf{Z}_{n-2}$, respectively. For n even (odd) these would give even (odd) permutations for $\Theta^n_{n-3,0}, \Theta^n_{n-3,1}$, respectively. Hence the result. □

2.2. *On constructing a partition basis for D_G*

In any partition in S_{2n}, perform the operations of ignoring either the elements i' or the elements $\!_{,}i$ for $i \in \{1, 2, \ldots, n\}$. These may be viewed as sub-partitions of the nodes $\!_{,}i$ and i', which we call 'bottoms' and 'tops', respectively. The elements of S^G_{2n}, the partition basis of $D_G(Q)$ can be constructed diagrammatically, by figuring out the possible 'top' and 'bottom' configurations that can be achieved by drawing connectivities on $G \times A_k$ for k large, and the possible ways of gluing the top and bottom by the $\#^p(z) = i$ 'propagating lines'. The ways of joining bottom to top are dictated by H^i_G, so the next step in this program is to determine the set of allowed tops, the bottoms being isomorphic under up–down transposition.

Proposition 6. For $G = \Theta^n_{p,q}$, $(q \leqslant p < n-1 < 2p+q)$, a partition basis element $z \in \Delta_i$ $(i < n)$ is also in the partition basis $\boldsymbol{B}^G_i$ of a $D_G(Q)\boldsymbol{E}_i$-module iff

(i) $\exists$ at least one part of z of the form (a') (a singleton node, $a \in \Lambda^0_G$), or

(ii) in the sub-partition of z consisting of nodes i', $i \in \Lambda_G^0$, one of the parts is of the form $(\cdots a'b' \cdots)$, where $(a,b) \in \Lambda_G^1$,
and the rest may be partitioned in any arbitrary way.
For all elements of the form (i), the configurations of 'tops' in $\boldsymbol{B}_i^G$ depend only on n and not on p, q.

Proof. (Only if.) Every word except 1 in D_G must begin with either $A^{j\cdot}$, $j \in \Lambda_G^0$ or $A^{i,j}$, $(i,j) \in \Lambda_G^1$.

(If.) (By construction of such a z.) Without loss of generality, let the partition z have the primed nodes in a sub-partition of the shape $(l_1, l_2, \ldots, l_r, 1)$ where the last part is the singleton (a') as in case (i). For case (ii), all words may be written as $A^{a,b}z$, with z constructed as in case (i). For $k = 1, 2, \ldots i$, the parts are $(a_1^{(k)}\ a_2^{(k)} \ldots a_{l_k}^{(k)} \pi(\c{k}))$, where $a_j^{(k)}$, $j = 1, \ldots, l_k$ are the primed nodes and $\pi(\c{k})$ is the image (under $\pi \in H_G^i$) of the bottom node $\c{k}$. For parts numbered $k = i+1, i+2, \ldots, r$, the parts are of the form $(a_1^{(k)}\ a_2^{(k)}\ \ldots\ a_{l_k}^{(k)})$, the $(r+1)$th part is the singleton (a'), and the remaining parts consist of singleton bottom nodes.

For all $\alpha, \beta \in \Lambda_G^0$ such that $|\alpha| = j = |\beta|$, let $\bar{\Phi}_j := \cup_{\alpha,\beta} \Phi_\alpha^\beta$. For a given z, define the words

$$\varphi_j^{(k)} \in \bar{\Phi}_{s(z,j,k)} \qquad (1 \leqslant k \leqslant r, 1 \leqslant j \leqslant l_k - 1)$$

where

$$s(z,j,k) = \sum_{u=1}^{k-1} l_u + j - \min(i, k-1)$$

and $\psi_j \in \bar{\Phi}_{l_j}$ by

$$\begin{aligned}
&\varphi_1^{(k)}(a_1) = n \qquad 1 \leqslant k \leqslant r \\
&\varphi_j^{(k)}(a_{j+1}^{(k)}) = 1 \qquad \forall j = 1, 2, \ldots l_k - 1 \qquad 1 \leqslant k \leqslant r \\
&\varphi_j^{(k)}(1) = n \qquad \forall j = 2, 3, \ldots, l_k - 1 \qquad 1 \leqslant k \leqslant r \\
&\varphi_j^{(k)}(\pi(j)) = \pi(\c{j}) \qquad k > j \\
&\psi_k(1) = \pi(\c{k}) \qquad k = 1, 2, \ldots, i
\end{aligned} \tag{26}$$

where the domain and ranges of the elements of $\bar{\Phi}_{s(z,j,k)}$, $\bar{\Phi}_{l_j}$ are nodes of G. Note that, $\pi(\c{k})$ indicates the positions of the bottom nodes in z as required. Such a construction is possible by corollary 4.1. The word

$$A^{a\cdot}\left(\prod_{k=1}^{i}\left[\left(\prod_{j=1}^{l_k-1}\varphi_j^{(k)}A^{1,n}A^{n\cdot}\right)\psi_k\right]\prod_{k=i+1}^{r}\left[\left(\prod_{j=1}^{l_k-1}\varphi_j^{(k)}A^{1,n}A^{n\cdot}\right)A^{1\cdot}\right]\right)\boldsymbol{E}_i$$

in D_G constructs the required z (see figure 7). □

This proposition will then enable us to estimate the cardinality of the partition basis of $D_G(Q)$, which we need for the next subsection.

2.3. *On locating the exceptional values of Q*

The next stage in giving the generic structure is to give the dimensions of the irreducibles and an explicit construction of a basis for each. Let ϵ_γ be a primitive idempotent of H_G^i, i.e. such that

$$R_\gamma = H_G^i \epsilon_\gamma \qquad \gamma \in \Gamma_G^i$$

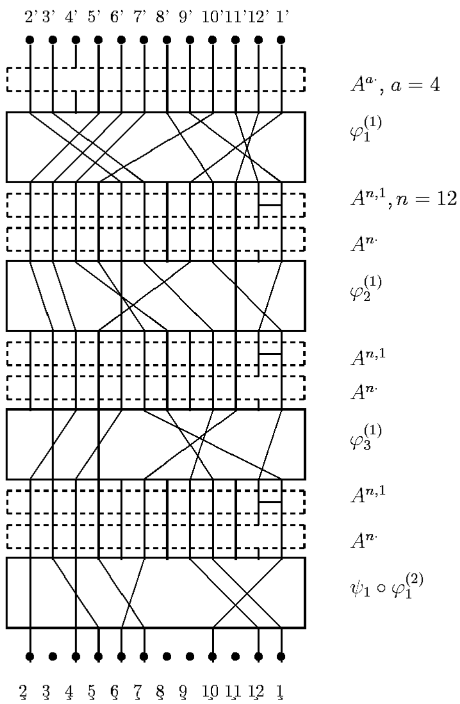

Figure 7. The figure depicts how the word $A^{4\cdot}(\prod_{j=1}^{3}\varphi_j^{(1)}A^{12,1}A^{12\cdot})\ \psi_1$ builds the partition ((8′ 9′ 10′ 11′ $\underset{,}{3}$)(12′ $\underset{,}{6}$) (7′ $\underset{,}{1}$)(6′ $\underset{,}{2}$)(5′ $\underset{,}{5}$)(3′ $\underset{,}{12}$)(2′ $\underset{,}{4}$)(1′ $\underset{,}{7}$)(4′)($\underset{,}{8}$)($\underset{,}{9}$)($\underset{,}{10}$)($\underset{,}{11}$)) on the lattice G $\times A_k$, where G is of the type $\Theta_{p,q}^{n=12}$. The broken boxes denote specific letters in D_G while those drawn with full lines are words in D_G. Note that the number of lines in the interior of each of the solid boxes above decreases from top to bottom.

is an irreducible representation of H_G^i (all of these are known, from [22] for example). Then

$$W_\gamma = D_G(\boldsymbol{E}_i \otimes \epsilon_\gamma)$$

builds generic irreducible W_γ.

In fact our present concern is not to determine the generic irreducible dimensions for a given G, but to locate the exceptional Q values (by analogy with the Beraha numbers for $G = A_n$ which is relevant for the two-dimensional case). To this end it is highly indicative to proceed as follows.

We first decide on a sequence approaching the large graph limit (not the same as the thermodynamic limit, see below). Thus, we take a sequence of graphs

$$G^{(-)} = \{G^j \ : \ j = 1, 2, \ldots\}$$

(with, say, $A_l \times A_m$ (l, m large) at the 'end' if we were to consider graphs appropriate for cubic lattice Potts models as in the next section) and then determine

$$k_{G^{(-)}}^{i,\gamma} = \lim_{j\to\infty} \frac{\dim(D_{G^{(j)}}\boldsymbol{E}_i^{(n_j)}\epsilon_\gamma)}{\dim(D_{G^{(j-1)}}\boldsymbol{E}_i^{(n_{j-1})}\epsilon_\gamma)} \qquad n_j = |\Lambda_{G^{(j)}}^0|$$

if it exists.

Since for the Q-state Potts model representation,

$$\frac{\dim(\rho_{G^{(j)}})}{\dim(\rho_{G^{(j-1)}})} = Q^m$$

for any sequence such that $|\Lambda^0_{G^{(j)}}| = |\Lambda^0_{G^{(j-1)}}| + m$, m a positive integer, if $k^{i,\gamma}_{G^{(-)}} > Q^m$, then Q is exceptional by the following argument [9]. Since $\boldsymbol{E}_0$ is a primitive idempotent (i.e., $\boldsymbol{E}_0 D_G \boldsymbol{E}_0 = \mathbb{C}\boldsymbol{E}_0$), $D_G\boldsymbol{E}_0$ is indecomposable. If $D_G(Q)$ were semisimple, $D_G\boldsymbol{E}_0$

would be contained in ρ_G with multiplicity 1 for all G. Thus, the evaluation of the case $\kappa_G := k^{0,(0)}_{G^{(-)}}$ is sufficient for any sequence $G^{(-)}$. So, if dim($D_G\boldsymbol{E}_0$)$> Q^n$ (for Q an integer) for $n = |\Lambda^0_G|$, Q is exceptional. For example $k^{i,\gamma}_{A^{(-)}} = 4$ where $A^{(-)} = \{G^j = A_j\}_{j\geqslant 1}$, and this signals the special nature of the $Q = 1, 2, 3$ state Potts models in two dimensions.

We can now consider the asymptotic growth rate of dimensions of the irreducible representations. In particular, we estimate a lower bound on the dimension of the module $D_G\boldsymbol{E}_0$.

Proposition 7. For $G = \Theta^n_{p,q}$ ($q \leqslant p \leqslant n-2$), and any $k \in \Re$, there exists a natural number M, such that $\dim(D_G\boldsymbol{E}_0) > k^n$ for $n > M$.

Proof. Let b_m be the number of ways of pairing m nodes (for some even number m). For any node (say 1), its partner (in the partition of shape $(2^{m/2})$) can be chosen in $m-1$ ways, while the rest of the pairs can be chosen in b_{m-2} ways. This determines $b_m = (m-1)b_{m-2}$ for all even m, with $b_1 = 1$. Thus for any $k \in \Re$, $\exists M$, a natural number, such that $b_m > k^m$ for $m > M$. From proposition 5, for any basis element of D_G for $G = \Theta^n_{p,q}$, $n-1$ primed nodes may be partitioned in any arbitrary way. The number of such possibilities is clearly larger that b_{n-1} (where we have chosen n to be odd, without loss of generality). Therefore,

$$\dim\,(D_G\boldsymbol{E}_0) > k^n. \qquad \square$$

Thus, for any integer Q and G of the type $\Theta^n_{p,q}$, the dimension of $D_G\boldsymbol{E}_0$ is larger than that of the Potts representation, ρ_G ($> Q^n$ for Q a positive integer).

We thus have

Proposition 8. Consider a sequence $G^{(-)} := \{G^{(j)}\}_{j\geqslant 1}$ of unsplitting graphs (except $G^{(j)} = \hat{A}_j$). For any positive integer Q, $\exists$ an integer n_1 s.t. $\forall n_2 > n_1$, $D_{G^{(n_2)}}$ is not semisimple.

3. Cubic lattice Potts models: $G = A_l \times A_m$

This is the case we are most interested in, to which we shall apply the results obtained in section 2.

Proposition 9. For $G = A_l \times A_m$, ($l, m \geqslant 2$), and $n = lm$,
(i) $H^{n-1}_G = \mathbf{A}(n-1)$
(ii) $H^i_G = \mathbf{S}(i), i < n-1$.

Proof. (i) For n even, G has a subgraph $\Theta^n_{n-3,0}$, and for n odd, it has a subgraph $\Theta^n_{n-3,1}$. Recall corollary 4.2. (ii) It is easy to see that $H^{n-2}_{\Theta^n_{n-3,0}}$ and $H^{n-2}_{\Theta^n_{n-3,1}}$ for $n = 4$ are isomorphic to each other and to $H^{n-1}_{\Theta^n_{n-2,0}}$ for $n = 3$. For $i \leqslant n-3$, recall (24). $\square$

Recall that the representations of $\mathbf{A}(j)$ are indexed by unordered pairs of partitions, $\lambda \vdash j$ and its conjugate λ', for $\lambda \neq \lambda'$. For $\lambda = \lambda' = \bar{\lambda}$, there are two non-isomorphic outer automorphism-conjugate representations labelled $\bar{\lambda}$ and $\bar{\lambda}^*$.

Corollary 9.1. For $G = A_l \times A_m$,

$$\Gamma_G = \cup_{i=0}^{n-2}\{\lambda \vdash i\} \cup \{(\lambda, \lambda'), \bar{\lambda}, \bar{\lambda}^* | \lambda, \bar{\lambda} \vdash n-1\} \cup \{\lambda \vdash n = (n)\}\,.$$

is the index set for irreducible representations of $D_G(Q)$.

Note that by filling in one of the diagonals of an elementary plaquette of $G = A_l \times A_m$, we obtain a graph which has $\Theta^n_{n-2,0}$ as a subgraph. The index set for the irreducible representations of the diagram algebras for such graphs with $\Theta^n_{n-2,0}$ subgraphs is the same as that of the complete graph on n nodes (see [20]).

Since $A_l \times A_m \supset \Theta^n_{p,q}$ for some p, q, we have (from proposition 5),

Corollary 9.2. For $G = A_l \times A_m$ $(n = lm)$, a partition basis element $z \in \Delta_i$ $(i < n-1)$ is also in the partition basis $\boldsymbol{B}_i^G$ of a $D_G(Q)\boldsymbol{E}_i$-module iff
(i) $\exists$ at least one part of z of the form (a') (a singleton node, $a \in \Lambda^0_G$), or
(ii) in the sub-partition of z consisting of nodes i', $i \in \Lambda^0_G$, one of the parts is of the form $(\cdots a'b' \cdots)$, where $(a, b) \in \Lambda^1_G$,
and the rest may be partitioned in any arbitrary way.

Also, since $G = A_l \times A_m$ is of the unsplitting type, we infer

Proposition 10. Consider a sequence $G^{(-)} := \{G^{(j)}\}_{j \in \mathbf{Z}_+}$, where $G^{(j)} = A_l \times A_m, l, m > 1, lm = j$. For any positive integer Q, $\exists$ an integer n_1 s.t. $\forall n_2 > n_1$, $D_{G^{(n_2)}}$ is not semisimple.

4. Discussion

For $G = A_l \times A_2$ for instance, it might have naively been expected that for large l, the results of the familiar A_l case might be approached. However, instead of the known growth rate of dimensions, i.e. 4, we get an unboundedly large number. This discrepancy might be attributed to the length, k, of the graph A_k in the transfer ('time-like') direction of the lattice $L = G \times A_k$, on which the partition function is evaluated. The connectivities of nodes are achieved involve 'permuting the nodes', i.e. the action of (20), where each shift (18) can only be realized for $k > \mu_i$. A restriction of the maximum k allowed will obviously reduce the dimensions and their growth rate κ_G. This is clearly necessary to define the true thermodynamic limit, where the volume has to increase in a specified fashion, keeping the ratios of lengths in all the directions of the lattice, L, fixed to some finite value, unlike in the definition of κ_G, where the size of the G was increased independent of k. In the two-dimensional case, the connectivities can all be achieved on $L = A_n \times A_k$ for $k \sim n$, and the problem does not arise. Thus, it might be useful to define a certain 'cut-off' height k of the representations of $D_G(Q)$ to narrow in on the physically relevant sectors of the representation theory.

We have indicated that for the smallest deviations away from chain graphs, e.g. $\Theta^n_{n-2,0}$, the diagram algebra is too large to carry directly useful physical information. Suitable quotients have to be implemented to reduce the size of the representations and an appropriately quotiented algebra would then be the analogous 'generic' algebra for the cubic lattice models. The special values of Q for which the algebra ceases to be semisimple is the obvious place to look for the quotient relations that are relevant for the Potts representation which is defined for integer values of Q. These integers are certainly a subset of the special points where the algebra ceases to be semisimple, as we have shown. We expect that the techniques outlined in the appendix can be extended to obtain the degeneracies of the cubic lattice Potts spectrum, which we would like to report in the future. We have also undertaken preliminary calculations on the location of other Q-values for which $D_G(Q)$ becomes non-semisimple for $G = A_l \times A_m$, and so far found only rationals. Further studies are in progress.

Acknowledgments

The authors would like to thank the EPSRC for financial support in the form of the grants GRJ25758 and GRJ29923. PM would also like to thank the Nuffield Foundation for partial financial support.

Appendix

Proposition 11. As left $D_G(Q)$ modules, $P_n\boldsymbol{E}_i \cong D_G\boldsymbol{E}_i \oplus R_i^G$ modulo $D_G\boldsymbol{E}_{i-1}D_G$ for any G, $|G| = n$, where R_i^G is either empty or a direct sum of θ_i copies of the trivial H_G^i-modules, $\chi_{(i)}$, $(i) \vdash i$, $R_i^G \cong \theta_i\chi_{(i)}$. Further, for $\lambda \vdash i \in \{0, 1, 2, \ldots, n\}$, $n = |\Lambda_G^0|$,

$$\mathrm{Res}^{P_G}_{D_G}\ {}_{P_G}V_\lambda = \bigoplus \begin{cases} {}_{D_G}W_\lambda \oplus \theta_i\chi_{(i)} & \forall i \in \{1, 2, \ldots, n-2\} \\ \begin{cases} {}_{D_G}W_{\lambda,\lambda'} \oplus \theta_i\chi_{(i)} & \lambda, \lambda' \vdash i = n-1, \\ {}_{D_G}W_\lambda \oplus {}_{D_G}W_\lambda^* \oplus \theta_i\chi_{(i)} & \text{for} \lambda = \lambda' \vdash i = n-1 \end{cases} \\ \oplus\theta_i\chi_{(i)} & \text{for} \quad i = n\,. \end{cases} \tag{A1}$$

Proof. $A^{j\cdot}P_n\boldsymbol{E}_i = A^{k,l}P_n\boldsymbol{E}_i = 0 \bmod D_G\boldsymbol{E}_i$, therefore,$D_Gx = 1x \bmod D_G\boldsymbol{E}_i \forall x \in P_n\boldsymbol{E}_i$. H_G^i acts trivially. For $\lambda \vdash n-1$, the labels of the representations of $P_n(Q)$ and $D_G(Q)$ are those of $\mathbf{S}(n-1)$ and $\mathbf{A}(n-1)$, respectively, and $\mathrm{Res}^{\mathbf{S}(n-1)}_{\mathbf{A}(n-1)}$ must be invoked. □

To characterize the generic structure of the algebra completely, it is necessary to determine the dimensions of its irreducible representations. Also, it is useful to characterize the inclusion of algebras, while approaching the large graph limit described above, in order to identify the subspaces that carry the information relevant for a physical interpretation. A preliminary step would be to determine how, for $H \subset G$, $D_G(Q)$-modules split up as $D_H(Q)$-submodules. Henceforth, we shall denote a left R-module M as ${}_RM$.

Proposition 12. For $G = A_l \times A_m$ and $G \supset H = A_{l-1} \times A_m$:

$$D_G\boldsymbol{E}_i \cong \oplus_{j=-m}^{m}(D_H\boldsymbol{E}_{i+j} \oplus R_{i+j}^{G,H})$$

as left $D_H(Q)$ modules, where $R_{i+j}^{G,H}$ is either empty or $\theta_{i+j}\chi_{(i+j)}$, where θ_k is the multiplicity of the trivial H_H^k-module, $\chi_{(k)}$, $(k) \vdash k$.

Proof. Let $\underline{m} := \{n-m+1, n-m+2, \ldots, n\}$ and $p_{\underline{m}} := \{1, 2, \ldots, n-m\}$. If $w \in D_G\boldsymbol{E}_i$ s.t. none of the parts of w is of the form $(\ldots k' \ldots l' \ldots)$ for $k' \in p_{\underline{m}}$ and $l' \in \underline{m}$, $D_Hw \cong \oplus_{j=0}^{m}D_H\boldsymbol{E}_{i-j}$. Each summand indexed by j denotes the number of parts of w which contain only primed nodes.

Similarly, if the nodes of $\underline{m}$ are in some $j \leqslant m$ parts with nodes of $p_{\underline{m}}$, we get $D_Hw \cong \oplus_{j=0}^{m}D_H\boldsymbol{E}_{i+m-j}$, where once again, j counts the number of parts of w containing only primed nodes.

As before, for $x \in D_G\boldsymbol{E}_i$, $D_Hx = 1x \bmod D_G\boldsymbol{E}_i$, and H_G^i thus acts trivially. □

Let ${}_{D_G}W_\gamma$ denote an irreducible left $D_G(Q)$ module, $\gamma \in \Gamma_G$. We are interested in the restriction $\mathrm{Res}^{D_G}_{D_H}\ {}_{D_G}W_\gamma$. Note the following inclusion of algebras:

$$\begin{array}{ccc} D_G(Q) & \subset & P_{|\Lambda_G^0|} \\ \cup & & \cup \\ D_H(Q) & \subset & P_{|\Lambda_H^0|} \end{array}$$

and consider the corresponding restrictions of modules:

$$\begin{array}{lll} \mathrm{Res}^{P_G}_{P_H}\ {}_{P_G}V_\lambda = \oplus_\mu \xi_{\lambda\mu}\ {}_{P_H}V_\mu & \lambda \in \mathcal{L}_{|\Lambda^0_G|} & \mu \in \mathcal{L}_{|\Lambda^0_H|} \\ \mathrm{Res}^{P_G}_{D_G}\ {}_{P_G}V_\lambda = \oplus_\gamma g_{\lambda\gamma}\ {}_{D_G}W_\gamma & \lambda \in \mathcal{L}_{|\Lambda^0_G|} & \gamma \in \Gamma_G \\ \mathrm{Res}^{P_H}_{D_H}\ {}_{P_H}V_\mu = \oplus_\eta h_{\mu\eta}\ {}_{D_H}W_\eta & \lambda \in \mathcal{L}_{|\Lambda^0_H|} & \mu \in \Gamma_H \\ \mathrm{Res}^{D_G}_{D_H}\ {}_{D_G}W_\gamma = \oplus_\mu m_{\gamma\eta}\ {}_{D_H}W_\eta & \gamma \in \Gamma_G & \eta \in \Gamma_H \end{array} \tag{A2}$$

where $P_G := P_{|\Lambda^0_G|}$ and (recall) the index sets labelling the irreducible representations of $P_n(Q)$ and $D_G(Q)$ are $\mathcal{L}_n$ and Γ_G, respectively.

Let $\dim({}_{D_G}W_\gamma)$ be denoted d_γ and $\dim({}_{D_H}W_\eta) := d_\eta$. Since the representations have already been assigned an index set, it is sufficient to determine the inclusion matrix $\mathcal{M}$, whose entries are the multiplicities $m_{\gamma\eta}$ in

$$d_\gamma = \sum_\eta m_{\gamma\eta} d_\eta$$

in order to complete the study of the generic irreducibles.

To obtain this, recall that $\mathrm{Res}^{P_G}_{P_H}$ is known, i.e. the coefficients $\xi_{\lambda\mu} \in \Xi^m$, where the inclusion matrix Ξ encodes the restriction information $P_{n-1}(Q) \subset P_n(Q)$ has been given in [19] and $m = |\Lambda^0_G| - |\Lambda^0_H|$. For a (left) $P_n(Q)$-module, ${}_{P_n}V_\lambda, \lambda \vdash i$,

$$\mathrm{Res}^{P_n}_{P_{n-1}}\ {}_{P_n}V_\lambda = \bigoplus \begin{cases} {}_{P_{n-1}}V_{\lambda'} & i-1 \dashv \lambda' \lhd \lambda \\ {}_{P_{n-1}}V_\lambda \oplus {}_{P_{n-1}}V_{\lambda'} & i \dashv \lambda' \rhd\!\lhd \lambda \\ {}_{P_{n-1}}V_{\lambda'} & i+1 \dashv \lambda' \rhd \lambda \end{cases}$$

where $\lambda \rhd \mu$ denotes the 'removal of a box' from λ to produce μ, $\lambda \lhd \mu$, denotes the 'addition of a box' to λ to produce μ, and $\lambda \rhd\!\lhd \mu$ means that we first remove a box from μ to obtain some ν (say), and then add a box to ν to obtain λ. Addition and removal of boxes correspond to the induction and restriction rules for symmetric group representations, called the Pieri (or Littlewood–Richardson) rules. Also note that in the above, ${}_{P_{n-1}}V_\lambda \cong {}_{P_n}V_\lambda$.

This is the key piece of information which, together with proposition (A1), will indicate the way to obtain $m_{\gamma\eta}$. Let us evaluate $\mathrm{Res}^{P_G}_{D_H}$ in two ways, corresponding to the paths in the diagram indicating the inclusion of algebras above (restrictions are transitive).

$$\mathrm{Res}^{P_G}_{D_H} = \mathrm{Res}^{D_G}_{D_H}\mathrm{Res}^{P_G}_{D_G} = \mathrm{Res}^{P_H}_{D_H}\mathrm{Res}^{P_G}_{P_H}\,.$$

Thus, from one path we get,

$$\mathrm{Res}^{P_G}_{D_H}\ {}_{P_G}V_\lambda = \mathrm{Res}^{P_H}_{D_H}\big(\mathrm{Res}^{P_G}_{P_H}\ {}_{P_G}V_\lambda\big) = \oplus_\mu \oplus_\eta \xi_{\lambda\mu} h_{\mu\eta}\ {}_{D_H}W_\eta \tag{A3}$$

while from the other,

$$\mathrm{Res}^{P_G}_{D_H}\,{}_{P_G}V_\lambda = \mathrm{Res}^{D_G}_{D_H}\big(\mathrm{Res}^{P_G}_{D_G}\,{}_{P_G}V_\lambda\big) = \oplus_\gamma \oplus_\eta g_{\lambda\gamma} m_{\gamma\eta}\ {}_{D_H}W_\eta\,. \tag{A4}$$

Let the inclusion matrices Σ_i and Υ^j_i encode the restriction information $\mathrm{Res}^{\mathsf{A}(i)}_{\mathsf{S}(i)}$ and $\mathrm{Res}^{\mathsf{A}(j)}_{\mathsf{A}(i)}$, with matrix elements $(\Sigma_i)_{a,b} = \varsigma^{(i)}_{a,b}$, and $(\Upsilon^j_i)_{a,b} = \upsilon^{(i,j)}_{a,b}$, respectively. Then, the restriction information between representations that are not among the list of one-dimensional representations $(\chi_{(i)})$, is extracted from the above.

$$\tilde{m}_{\gamma\eta} = \begin{cases} \xi_{\gamma\eta}, \gamma \vdash k < n-1 & \eta \vdash l < n-m-1 \\ \displaystyle\sum_\mu \xi_{\gamma\mu}\varsigma^{(n-m-1)}_{\mu\eta} & \gamma \vdash k < n-1, \quad \mu, \eta \vdash l = n-m-1 \\ \upsilon^{(n-1,n-m-1)}_{\gamma\eta} & \gamma \vdash n-1,\ \eta \vdash n-m-1\,. \end{cases}$$

In the above, we have used $\tilde{m}_{\gamma\eta}$ instead of $m_{\gamma\eta}$ to indicate that $\tilde{m}_{\gamma\eta}$ does not give the multiplicities θ_i of the one-dimensional representations, $\chi_{(i)}$. The number of such $\chi_{(i)}$ is not known in general. Diagrammatically their determination is a combinatorial problem of enumerating the number of 'top' configurations that are characterized by corollary 9.2.

We have constructed an algorithm for their enumeration by using recurrence relations for $G = A_l \times A_2$, but we have not been able to solve it in closed form. For arbitrary rectangular graphs, the combinatorics is much more complicated.

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